(24) $A=\begin{pmatrix}1&0&1\\0&2&-2\\0&0&3\end{pmatrix}$ Pro $\forall B\in M$ musí platit $A\cdot B=B\cdot A$, neboli

$$\begin{pmatrix}1&0&1\\0&2&-2\\0&0&3\end{pmatrix}\begin{pmatrix}b_{11}&b_{12}&b_{13}\\b_{21}&b_{22}&b_{23}\\b_{31}&b_{32}&b_{33}\end{pmatrix}=\begin{pmatrix}b_{11}&b_{12}&b_{13}\\b_{21}&b_{22}&b_{23}\\b_{31}&b_{32}&b_{33}\end{pmatrix}\begin{pmatrix}1&0&1\\0&2&-2\\0&0&3\end{pmatrix}$$

Tvar matic komutujících s A:

$$B\in M\Rightarrow B=\begin{pmatrix}b_{11}&0&\frac{1}{2}(b_{33}-b_{11})\\0&b_{22}&2b_{22}-2b_{33}\\0&0&b_{33}\end{pmatrix}$$

$\Uparrow$

$$\begin{pmatrix}b_{11}+b_{31}&b_{12}+b_{32}&b_{13}+b_{33}\\2b_{21}-2b_{31}&2b_{22}-2b_{32}&2b_{23}-2b_{33}\\3b_{31}&3b_{32}&3b_{33}\end{pmatrix}=\begin{pmatrix}b_{11}&2b_{12}&b_{11}-2b_{12}+3b_{13}\\b_{21}&2b_{22}&b_{21}-2b_{22}+3b_{23}\\b_{31}&2b_{32}&b_{31}-2b_{32}+3b_{33}\end{pmatrix}$$

- $3b_{31}=b_{31}$
  $\underline{b_{31}=0}$
- $2b_{21}-3\overset{=0}{b_{31}}=b_{21}$
  $\underline{b_{21}=0}$
- $3b_{32}=2b_{32}$
  $\underline{b_{32}=0}$
- $b_{12}+\overset{=0}{b_{32}}=2b_{12}$
  $\underline{b_{12}=0}$
- $b_{11}+\overset{=0}{b_{31}}=b_{11}$
  $\underline{b_{11}\text{ je libovolné}}$
- $2b_{22}-2\overset{=0}{b_{32}}=2b_{22}$
  $\underline{b_{22}\text{ je libovolné}}$
- $3b_{33}=\overset{=0}{b_{31}}-2\overset{=0}{b_{32}}+3b_{33}$
  $\underline{b_{33}\text{ je libovolné}}$
- $b_{13}+b_{33}=b_{11}-2\overset{=0}{b_{12}}+3b_{13}$
  $b_{33}-b_{11}=2b_{13}$
  $\underline{b_{13}=\frac{1}{2}(b_{33}-b_{11})}$
- $2b_{23}-2b_{33}=\overset{=0}{b_{21}}-2b_{22}+3b_{23}$
  $\underline{2b_{22}-2b_{33}=b_{23}}$

- Z vlastností součinu matic víme, že s maticí A komutuje jednotková matice I_3 a matice A^2.

Nyní ověříme axiomy báze:

1) A, I_3 a A^2 musí být lineárně nezávislé, tj. $a\cdot A+b\cdot I_3+c\cdot A^2=0\Rightarrow a=b=c=0$

$$A^2=\begin{pmatrix}1&0&1\\0&2&-2\\0&0&3\end{pmatrix}\begin{pmatrix}1&0&1\\0&2&-2\\0&0&3\end{pmatrix}=\begin{pmatrix}1&0&4\\0&4&-10\\0&0&9\end{pmatrix}$$

$$a\cdot\begin{pmatrix}1&0&1\\0&2&-2\\0&0&3\end{pmatrix}+b\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}+c\begin{pmatrix}1&0&4\\0&4&-10\\0&0&9\end{pmatrix}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$$

$$\begin{pmatrix}a+b+c&0&a+4c\\0&2a+b+4c&-2a-10c\\0&0&3a+b+9c\end{pmatrix}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$$

$a+4c=0 \quad |\cdot 2$
$-2a-10c=0$
$2a+8c=0$
$-2a-10c=0$
$-2c=0$
$\underline{c=0}$

$a+0=0$
$\underline{a=0}$
$a+b+c=0$
$0+b+0=0$
$\underline{b=0}$

✓ Matice A, I_3 a A^2 jsou lineárně nezávislé

2) $\forall B\in M$, tedy $\forall B=\begin{pmatrix}b_{11}&0&\frac{1}{2}b_{33}-\frac{1}{2}b_{11}\\0&b_{22}&2b_{22}-2b_{33}\\0&0&b_{33}\end{pmatrix}$ musí být možné zapsat ve tvaru $a\cdot A+b\cdot I_3+c\cdot A^2$, tj.

$$\begin{pmatrix}a+b+c&0&a+4c\\0&2a+b+4c&-2a-10c\\0&0&3a+b+9c\end{pmatrix}=\begin{pmatrix}b_{11}&0&\frac{1}{2}b_{33}-\frac{1}{2}b_{11}\\0&b_{22}&2b_{22}-2b_{33}\\0&0&b_{33}\end{pmatrix}\Rightarrow$$

$$\begin{array}{ccc}a&b&c\end{array}$$

$$\left(\begin{array}{ccc|c}1&1&1&b_{11}\\1&0&4&\frac{1}{2}b_{33}-\frac{1}{2}b_{11}\\2&1&4&b_{22}\\-2&0&-10&2b_{22}-2b_{33}\\3&1&9&b_{33}\end{array}\right)\sim\left(\begin{array}{ccc|c}1&1&1&b_{11}\\0&-1&3&\frac{1}{2}b_{33}-\frac{3}{2}b_{11}\\0&-1&2&-2b_{11}+b_{22}\\0&1&-6&3b_{22}-2b_{33}\\0&-2&6&-3b_{11}+b_{33}\end{array}\right)\sim$$

$$\sim\left(\begin{array}{ccc|c}1&1&1&b_{11}\\0&-1&3&\frac{1}{2}b_{33}-\frac{3}{2}b_{11}\\0&0&-4&4b_{22}-2b_{11}-2b_{33}\\0&0&-3&3b_{22}-\frac{3}{2}b_{33}-\frac{3}{2}b_{11}\\0&0&0&0\end{array}\right)\sim\left(\begin{array}{ccc|c}1&1&1&b_{11}\\0&-1&3&\frac{1}{2}b_{33}-\frac{3}{2}b_{11}\\0&0&2&-2b_{22}+b_{11}+b_{33}\\0&0&0&0\\0&0&0&0\end{array}\right)$$

$\Rightarrow$ soustava má řešení $\Rightarrow \forall B\in M$ lze zapsat jako lin. komb. matic A, I_3 a $A^2\Rightarrow$ tyto matice tvoří bázi

$$\underline{\mathcal{B}_M=\left\{\begin{pmatrix}1&0&1\\0&2&-2\\0&0&3\end{pmatrix},\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\begin{pmatrix}1&0&4\\0&4&-10\\0&0&9\end{pmatrix}\right\}}$$

$\underline{\dim[M]=3}$ ... počet prvků báze